

Further results on the Hamilton-Waterloo problem

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Abstract

In this paper, we almost completely solve the existence of an almost resolvable cycle system with odd cycle length. We also use almost resolvable cycle systems as well as other combinatorial structures to give some new solutions to the Hamilton-Waterloo problem.

Key words: 2-factorization; Almost resolvable cycle system; Hamilton-Waterloo Problem

1 Introduction

In this paper, the vertex set and the edge set of a graph H will be denoted by $V(H)$ and $E(H)$, respectively. We denote the cycle of length k by k -cycle or C_k , the complete graph on v vertices by K_v and denote the complete u -partite graph with u parts of size g by $K_u[g]$.

A *factor* of a graph H is a spanning subgraph of H . Suppose G is a subgraph of a graph H , a G -*factor* of graph H is a set of edge-disjoint subgraphs of H , each isomorphic to G . And a G -*factorization* of H is a set of edge-disjoint G -factors of H . A C_k -factorization of a graph H is a partition of $E(H)$ into C_k -factors. In [4, 6, 20, 23, 24, 25, 33, 34], we can obtain the following result.

Theorem 1.1. *There exists a C_k -factorization of $K_u[g]$ if and only if $g(u-1) \equiv 0 \pmod{2}$, $gu \equiv 0 \pmod{k}$, k is even when $u = 2$, and $(k, u, g) \notin \{(3, 3, 2), (3, 6, 2), (3, 3, 6), (6, 2, 6)\}$.*

An r -*factor* is a factor which is r -regular. It's obvious that a 2-factor consists of a collection of disjoint cycles. A 2-*factorization* of a graph H is a partition of $E(H)$ into 2-factors. The well-known *Hamilton-Waterloo problem* is the problem of determining whether K_v (for v odd) or K_v minus a 1-factor (for v even) has a 2-factorization in which there are exactly α C_m -factors and β C_n -factors. The authors [39] generalize this problem to a general graph H , and use $\text{HW}(H; m, n; \alpha, \beta)$ to denote a 2-factorization of H in which there are exactly α

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C_m -factors and β C_n -factors. So when $H = K_v$ (for v odd) or K_v minus a 1-factor (for v even), an $\text{HW}(H; m, n; \alpha, \beta)$ is a solution to the original Hamilton-Waterloo problem, denoted by $\text{HW}(v; m, n; \alpha, \beta)$. Further, denote by $\text{HWP}(v; m, n)$ the set of (α, β) for which a solution $\text{HW}(v; m, n; \alpha, \beta)$ exists. It is easy to see that the necessary conditions for the existence of an $\text{HW}(v; m, n; \alpha, \beta)$ are $m|v$ when $\alpha > 0$, $n|v$ when $\beta > 0$ and $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$. When $\alpha\beta = 0$, there is a solution to $\text{HW}(v; m, n; \alpha, \beta)$, see [4, 6, 20]. From now on, we suppose that $\alpha\beta \neq 0$.

A lot of work has been done for small values of m and n . A complete solution for the existence of an $\text{HW}(v; 3, n; \alpha, \beta)$ in the cases $n \in \{4, 5, 7\}$ is given in [1, 12, 26, 32, 39]. For the case $(m, n) \in \{(3, 15), (5, 15), (4, 6), (4, 8), (4, 16), (8, 16)\}$, see [1]. The existence of an $\text{HW}(v; 4, n; \alpha, \beta)$ for odd $n \geq 3$ has been solved except possibly when $v = 8n$ and $\alpha = 2$, see [22, 32, 39]. It is shown in [21] that the necessary conditions for the existence of an $\text{HW}(v; 3, 9; \alpha, \beta)$ are also sufficient except possibly when $\beta = 1$. Many infinite classes of $\text{HW}(v; 3, 3x; \alpha, \beta)$ s are constructed in [5]. Much attention to the Hamilton-Waterloo problem has been dedicated to the case of triangle factors and Hamilton factors, the results for this case can be found in [13, 14, 19, 29]. In [27], the authors give a complete solution for the existence of an $\text{HW}(v; 4k, v; \alpha, \beta)$. Recently, Burgess et al. [8] have made significant progress on the Hamilton-Waterloo problem for uniform odd cycle factors.

Theorem 1.2. *If m, n and t are odd integers with $n \geq m \geq 3$, then $(\alpha, \beta) \in \text{HWP}(mnt; m, n)$ if and only if $\alpha, \beta \geq 0$ and $\alpha + \beta = \frac{mnt-1}{2}$, except possibly when:*

- $t > 1$ and $\beta = 1$ or 3 , or $(m, n, \beta) = (5, 9, 5), (5, 9, 7), (7, 9, 5), (7, 9, 7), (3, 13, 5)$;
- $t = 1$ and $\beta \in [1, 2, \dots, \frac{n-3}{2}] \cup \{\frac{n+1}{2}, \frac{n+5}{2}\}$, $(m, \alpha) = (3, 2), (3, 4)$ or $(m, n, \alpha, \beta) = (3, 11, 6, 10), (3, 13, 8, 11), (5, 7, 9, 8), (5, 9, 11, 11), (5, 9, 13, 9), (7, 9, 20, 11)$ or $(7, 9, 22, 9)$.

We continue to consider this problem, the final main results will be the following.

Theorem 1.3. *If m, n and t are odd integers with $n \geq m \geq 3$, then $(\alpha, \beta) \in \text{HWP}(mnt; m, n)$ if and only if $\alpha, \beta \geq 0$ and $\alpha + \beta = \frac{mnt-1}{2}$, except possibly when:*

- $t > 1$ and $\beta = 1$ or 3 ;
- $t = 1$ and $\beta \in [1, 2, \dots, \frac{n-3}{2}] \cup \{\frac{n+1}{2}, \frac{n+5}{2}\}$, $(m, \alpha) = (3, 2), (3, 4)$.

When $m = k$ and $n = 2kt + 1$, we can also get the following results.

Theorem 1.4. *Let $J = \{4, 6\} \cup [8, 9, \dots, k(k-1)t + \frac{k-3}{2}]$. If $t \geq 1$ and odd integer $k \geq 3$, then $(\alpha, \beta) \in \text{HWP}(k(2kt+1); k, 2kt+1)$ in the following four cases:*

- (1) $k = 3$: $t \neq 1, 2$, $\beta \in [4, 5, \dots, 6t-6] \cup \{6t\}$ is even;
- (2) $k = 5$: $\beta \in J \setminus \{20t-3, 20t-1\}$;
- (3) $k = 7, 9$: $\beta \in J$;
- (4) $k \geq 11$: $t \neq 2$, $\beta \in J$.

Theorem 1.5. *If $v \equiv 0 \pmod{4kt}$, then $(\alpha, \beta) \in \text{HWP}(v; 4k, 4kt)$ if and only if $k \geq 1$, $t \geq 2$, $\alpha, \beta \geq 0$ and $\alpha + \beta = \frac{v-2}{2}$.*

In Section 2, we will introduce the definition of an almost resolvable k -cycle system, then we give some constructions for almost resolvable cycle systems with odd orders. In Section 3, we will use almost resolvable k -cycle system as well as other combinatorial structures to give some new recursive constructions for $\text{HW}(v; m, n; \alpha, \beta)$ with $m \equiv n \equiv 1 \pmod{2}$. In the last section, we shall present some direct constructions and use these recursive constructions in Section 3 to prove our main results.

2 Almost resolvable k -cycle systems

A k -cycle system of order v is that a collection of k -cycles which partition the edges of K_v . A k -cycle system of order v exists if and only if $3 \leq k \leq v$, $v \equiv 1 \pmod{2}$ and $v(v-1) \equiv 0 \pmod{2k}$ [3, 36]. A k -cycle system of order v is *resolvable* if it has a C_k -factorization. By Theorem 1.1, a resolvable k -cycle system of order v exists if and only if $3 \leq k \leq v$, v and k are odd, and $v \equiv 0 \pmod{k}$. If $v \equiv 1 \pmod{2k}$, then the k -cycle system is not resolvable. In this case, Vanstone et al. [37] started the research of the existence of an almost resolvable k -cycle system.

A collection of $(v-1)/k$ disjoint k -cycles is called an *almost parallel class*. In a k -cycle system of order $v \equiv 1 \pmod{2k}$, the maximum possible number of almost parallel classes is $(v-1)/2$ in which case a *half-parallel class* containing $(v-1)/2k$ disjoint k -cycles is left over. A k -cycle system of order v whose cycle set can be partitioned into $(v-1)/2$ almost parallel classes and a half-parallel class is called an *almost resolvable k -cycle system*, denoted by $k\text{-ARCS}(v)$.

For recursive constructions of almost resolvable k -cycle systems, C. C. Lindner, et al. [28] have considered the general existence problem of almost resolvable k -cycle system from the commutative quasigroup for $k \equiv 0 \pmod{2}$ and make a hypothesis: if there exists a $k\text{-ARCS}(2k+1)$ for $k \equiv 0 \pmod{2}$ and $k \geq 8$, then there exists a $k\text{-ARCS}(2kt+1)$ except possibly for $t = 2$. H. Cao et al. [11, 31] continue to consider the recursive constructions of an almost resolvable k -cycle system for $k \equiv 1 \pmod{2}$. By using recursive method and direct constructions, some classes of almost resolvable cycle systems with small orders have been obtained. The known results on the existence of an almost resolvable cycle system of order n are summarized as below.

Theorem 2.1. ([2, 10, 11, 15, 16, 28, 37]) *Let $k \geq 3$, $t \geq 1$ be integers and $n = 2kt+1$. There exists a $k\text{-ARCS}(n)$ for $k \in \{3, 4, 5, 6, 7, 8, 9, 10, 14\}$, except for $(k, n) \in \{(3, 7), (3, 13), (4, 9)\}$ and except possibly for $(k, n) \in \{(8, 33), (14, 57)\}$.*

Theorem 2.2. ([31]) *There exists a k -ARCS($2kt+1$) for $t \geq 1$, $11 \leq k \leq 49$, $k \equiv 1 \pmod{2}$ and $t \neq 2, 3, 5$.*

In this section we focus on constructions of almost resolvable cycle system with odd orders. The main idea is to find some initial cycles with special properties such that all the required almost parallel classes can be obtained from them. We also need the following notions for our constructions.

Suppose Γ is an additive group and $I = \{\infty\}$ is a set which is disjoint with Γ . We will consider an action of Γ on $\Gamma \cup I$ which coincides with the *right regular action* on the elements of Γ , and the action of Γ on I will coincide with the identity map. Given a graph H with vertices in $\Gamma \cup I$, the *translate* of H by an element γ of Γ is the graph $H + \gamma$ obtained from H by replacing each vertex $x \in V(H)$ with the vertex $x + \gamma$. The *development* of H under a subgroup Σ of additive group Γ is the collection $dev_{\Sigma}(H) = \{H + x \mid x \in \Sigma\}$ of all translates of H by an element of Σ . The *list of differences* of a graph H with vertices in Γ is the multiset ΔH of all possible differences $x - y$ with (x, y) an ordered pair of adjacent vertices of H .

2.1 Constructions for k -ARCS($2k+1$)

For our constructions, we suppose $\Gamma = Z_u \times Z_2$ and $I = \{\infty\}$. For a graph H with vertices in $\Gamma \cup I$ and any pair $(j, j') \in Z_2 \times Z_2$, we define *list of (j, j') -differences of H* as the multiset $\Delta_{(j, j')}$ of all possible differences $x - y \in Z_u$ with (x, j) adjacent to (y, j') in H .

Lemma 2.3. *Let $v = 2k + 1$ and let F be a vertex-disjoint union of two cycles of length k satisfying the following conditions:*

- (i) $V(F) = (Z_k \times Z_2) \cup \{\infty\} \setminus \{(a, b)\}$, $(a, b) \in Z_k \times Z_2$;
- (ii) ∞ has a neighbor in $Z_k \times \{0\}$ and the other neighbor in $Z_k \times \{1\}$;
- (iii) $\Delta_{(0,0)}F \supseteq Z_k \setminus \{0\}$, $\Delta_{(0,1)}F \supseteq Z_k$, $\Delta_{(1,1)}F \supseteq Z_k \setminus \{0, \pm d\}$, $(d, k) = 1$.

Then, there exists a k -ARCS($2k+1$).

Proof: Let $V(K_v) = (Z_k \times Z_2) \cup \{\infty\}$. The half parallel class is the cycle $C_0 = ((0, 1), (d, 1), (2d, 1), \dots, (kd - d, 1))$ since $(d, k) = 1$. By (i), we know that F is an almost parallel class. All the required k almost parallel classes will be generated from F by $(+1 \pmod{k}, -)$.

Now we show that the half parallel class and the k almost parallel classes form a k -ARCS($2k+1$). Let $F' = \{(a, 1), (a + d, 1) \mid a \in Z_k\}$ and $\Sigma := Z_k \times \{0\}$. Let $\mathcal{F} = dev_{\Sigma}(F) \cup \{F'\}$. The total number of edges-counted with their respective multiplicities-covered by the almost parallel classes and half parallel class of \mathcal{F} is $k(2k+1)$, that is exactly the size of $E(K_v)$. Therefore, we only need to prove that every pair of vertices lies in a suitable translate of F or in F' . By (ii), an edge $\{(z, j), \infty\}$ of K_v must appear in a cycle of $dev_{\Sigma}(F)$.

Now consider an edge $\{(z, j), (z', j')\}$ of K_v whose vertices both belong to $Z_k \times Z_2$. If $j = j' = 1$ and $z - z' \in \{\pm d\}$, then this edge belongs to F' . In all other cases there is, by (iii), an edge of F of the form $\{(w, j), (w', j')\}$ such that $w - w' = z - z'$. It then follows that $F + (-w' + z', 0)$ is an almost parallel class of $\text{dev}_\Sigma(F)$ containing the edge $\{(z, j), (z', j')\}$ and the assertion follows. \square

Here we use Lemma 2.3 to give a construction of a k -ARCS($2k + 1$) for any odd $k \geq 9$.

Lemma 2.4. *For any odd $k \geq 9$, there exists a k -ARCS($2k + 1$).*

Proof: Let the vertex set be $(Z_k \times Z_2) \cup \{\infty\}$ and $(a, b) = (0, 1)$. For the element d and two initial cycles C_1 and C_2 in F , we distinguish in two cases $k \equiv 1 \pmod{4}$ and $k \equiv 3 \pmod{4}$.

(1) $k \equiv 1 \pmod{4}$. Let $k = 4n + 1$ and $d = 2n$.

The cycle C_1 is the concatenation of the sequences T_1 , $(0, 0)$ and T_2 defined as follows:

$$T_1 = ((n, 0), (-n, 1), \dots, (n - i, 0), \underline{(-(n - i), 1)}, \dots, (1, 0), (-1, 1));$$

$$T_2 = ((1, 1), (-1, 0), \dots, \underline{(1 + i, 1)}, \underline{(-(1 + i), 0)}, \dots, (n, 1), (-n, 0)).$$

The cycle C_2 is the concatenation of the sequences ∞ , T_1 and T_2 , where

$$T_1 = ((-2n, 0), (2n, 0), \dots, \underline{(-(2n - i), 0)}, \underline{(2n - i, 0)}, \dots, (-(n + 1), 0), (n + 1, 0));$$

$$T_2 = ((n + 1, 1), (-(n + 1), 1), \dots, \underline{(n + 1 + i, 1)}, \underline{(-(n + 1 + i), 1)}, \dots, (2n, 1), (-2n, 1)).$$

(2) $k \equiv 3 \pmod{4}$. Let $k = 4n + 3$ and $d = 2$.

The cycle C_1 is the concatenation of the sequences T_1 , T_2 and T_3 , where

$$T_1 = ((-2, 0), (2, 1), \dots, \underline{(-(2 + 2i), 0)}, \underline{(2 + 2i, 1)}, \dots, (-2n, 0), (2n, 1));$$

$$T_2 = ((2n, 0), (-2n, 1), \dots, \underline{(2n - 2i, 0)}, \underline{(-(2n - 2i), 1)}, \dots, (2, 0), (-2, 1));$$

$$T_3 = ((-1, 0), (1, 1), (0, 0)).$$

The cycle C_2 is the concatenation of the sequences ∞ , T_1 , T_2 and T_3 defined as follows:

$$T_1 = ((2n + 1, 0), (-(2n + 1), 0), \dots, \underline{(2n + 1 - 2i, 0)}, \underline{(-(2n + 1 - 2i), 0)}, \dots, (3, 0), (-3, 0));$$

$$T_2 = ((1, 0), (-1, 1));$$

$$T_3 = ((3, 1), (-3, 1), \dots, \underline{(3 + 2i, 1)}, \underline{(-(3 + 2i), 1)}, \dots, (2n + 1, 1), (-(2n + 1), 1)).$$

It is straightforward to check that F satisfies all the conditions of Lemma 2.3. For brevity, we don't list the vertices and differences in F . The proof is complete. \square

2.2 Constructions for k -ARCS($6k + 1$)

We construct a k -ARCS($6k + 1$) for any odd $k \geq 9$. We give the following lemma which is similar to Lemma 2.3.

Lemma 2.5. *Let $v = 6k + 1$ and let F be a vertex-disjoint union of six cycles of length k satisfying the following conditions:*

(i) $V(F) = (Z_{3k} \times Z_2) \cup \{\infty\} \setminus \{(a, b)\}$, $(a, b) \in Z_{3k} \times Z_2$;

(ii) ∞ has a neighbor in $Z_{3k} \times \{0\}$ and the other neighbor in $Z_{3k} \times \{1\}$;

(iii) $\Delta_{(0,0)}F \supseteq Z_{3k} \setminus \{0\}$, $\Delta_{(0,1)}F \supseteq Z_{3k}$, $\Delta_{(1,1)}F \supseteq Z_{3k} \setminus \{0, \pm 3\}$.

Then, there exists a k -ARCS($6k+1$).

Proof: Let $V(K_v) = (Z_{3k} \times Z_2) \cup \{\infty\}$. The half-parallel class is $\{C_0 + (i, 0) \mid i = 0, 1, 2\}$, where $C_0 = ((0, 1), (3, 1), (6, 1), \dots, (3(k-1), 1))$. F is an almost parallel class by (i). All the required $3k$ almost parallel classes will be generated from F by $(+1 \pmod{3k}, -)$.

Next, we show that the half parallel class and the $3k$ almost parallel classes form a k -ARCS($6k+1$). Let $F' = \{(a, 1), (a+3, 1) \mid a \in Z_{3k}\}$ and $\Sigma := Z_{3k} \times \{0\}$. Let $\mathcal{F} = \text{dev}_\Sigma(F) \cup \{F'\}$. The total number of edges-counted with their respective multiplicities-covered by the almost parallel classes and half parallel class of \mathcal{F} is $3k(6k+1)$, that is exactly the size of $E(K_v)$. So, we only need to prove that every pair of vertices lies in a suitable translate of F or in F' . By (ii), an edge $\{(z, j), \infty\}$ of K_v must appear in a cycle of $\text{dev}_\Sigma(F)$.

Next, we consider an edge $\{(z, j), (z', j')\}$ of K_v whose vertices both belong to $Z_{3k} \times Z_2$. If $j = j' = 1$ and $z - z' \in \{\pm 3\}$, this edge belongs to F' . In all other cases there is, by (iii), an edge of F of the form $\{(w, j), (w', j')\}$ such that $w - w' = z - z'$. It then follows that $F + (-w' + z', 0)$ is an almost parallel class of $\text{dev}_\Sigma(F)$ containing the edge $\{(z, j), (z', j')\}$. Hence, the assertion holds. \square

Lemma 2.6. *For any $k \geq 9$ and $k \equiv 1 \pmod{4}$, there exists a k -ARCS($6k+1$).*

Proof: Let the vertex set be $(Z_{3k} \times Z_2) \cup \{\infty\}$ and $(a, b) = (0, 1)$. Let $k = 4n + 1$, $n \geq 2$. The six initial cycles in F are listed as below.

First of all, we construct the following cycles C_i , $1 \leq i \leq 4$, where

$$C_1 = ((3n+1, 0), (-(3n+1), 0), \dots, \underline{(3n+1+i, 0), (-(3n+1+i), 0)}, \dots, (5n, 0), (-5n, 0), (5n+3, 0));$$

$$C_2 = ((3n+1, 1), (-(3n+1), 1), \dots, \underline{(3n+1+i, 1), (-(3n+1+i), 1)}, \dots, (5n, 1), (-5n, 1), (5n+3, 1));$$

$$C_3 = ((n+1, 0), (-(n+1), 1), \dots, \underline{(n+1+i, 0), (-(n+1+i), 1)}, \dots, (3n, 0), (-3n, 1), (-n, 0));$$

$$C_4 = ((n+1, 1), (-(n+1), 0), \dots, \underline{(n+1+i, 1), (-(n+1+i), 0)}, \dots, (3n, 1), (-3n, 0), (-n, 1)).$$

To construct C_5 and C_6 , we start with $k \in \{9, 13, 17\}$:

$$k = 9: \quad C_5 = (\infty, (2, 1), (1, 1), (-1, 1), (1, 0), (-13, 1), (0, 0), (-1, 0), (2, 0));$$

$$C_6 = ((-13, 0), (-11, 0), (11, 1), (12, 0), (-12, 1), (-12, 0), (12, 1), (11, 0), (-11, 1)).$$

$$k = 13: \quad C_5 = (\infty, (3, 1), (-2, 1), (2, 1), (1, 1), (-1, 1), (1, 0), (-19, 1), (0, 0), (-1, 0), (2, 0), (-2, 0), (3, 0));$$

$$C_6 = ((-19, 0), (-17, 0), (17, 1), (16, 0), (-16, 1), (-18, 0), (-18, 1), (17, 0), (-17, 1), (19, 0), (16, 1), (-16, 0), (19, 1)).$$

$$k = 17: \quad C_5 = (\infty, (4, 1), (-3, 1), (3, 1), (-2, 1), (2, 1), (1, 1), (-1, 1), (1, 0), (-25, 1), (0, 0), (-1, 0), (2, 0), (-2, 0), (3, 0), (-3, 0), (4, 0));$$

$$C_6 = ((-25, 0), (-23, 0), (22, 1), (-22, 0), (24, 1), (25, 0), (21, 1), (-21, 0), (-24, 1), (-24, 0), (-21, 1), (21, 0), (25, 1), (24, 0), (-22, 1), (22, 0), (-23, 1)).$$

For $k > 17$, the cycle C_5 is the concatenation of the sequences ∞ , T_1 , T_2 and T_3 , where $T_1 = ((n, 1), (-(n-1), 1), (n-1, 1), \dots, \underline{(-(n-1-i), 1), (n-1-i, 1)}, \dots, (-2, 1), (2, 1), (1, 1), (-1, 1));$

$$T_2 = ((1, 0), (-(6n+1), 1), (0, 0), (-1, 0));$$

$$T_3 = ((2, 0), (-2, 0), \dots, \underline{(2+i, 0), (-(2+i), 0)}, \dots, (n-1, 0), (-(n-1), 0), (n, 0)).$$

For the cycle C_6 , we consider the following three cases.

Case 1: $k \equiv 1 \pmod{12}$, $k \geq 25$.

The cycle C_6 is the concatenation of the sequences T_1, T_2, \dots, T_{12} defined as follows:

$$T_1 = ((-(6n+1), 0), (-(6n-1), 0));$$

$$T_2 = ((6n-5, 1), (-(6n-4), 0), \dots, \underline{(6n-5-3i, 1), (-(6n-4-3i), 0)}, \dots, (5n+4, 1), (-(5n+5), 0));$$

$$T_3 = ((5n+1, 1), (-(5n+1), 0));$$

$$T_4 = ((5n+6, 1), (-(5n+4), 0), \dots, \underline{(5n+6+3i, 1), (-(5n+4+3i), 0)}, \dots, (6n-3, 1), (-(6n-5), 0));$$

$$T_5 = ((6n+1, 1), (-(6n-2), 0), (6n-1, 1), (6n, 0), (-6n, 1), (6n-2, 0));$$

$$T_6 = ((-(6n-3), 1), (6n-4, 0), \dots, \underline{(-(6n-3-3i), 1), (6n-4-3i, 0)}, \dots, (-(5n+6), 1), (5n+5, 0));$$

$$T_7 = ((-(5n+3), 1), (5n+2, 0), (-(5n+2), 1), (-(5n+2), 0), (5n+2, 1), (-(5n+3), 0));$$

$$T_8 = ((5n+5, 1), (-(5n+6), 0), \dots, \underline{(5n+5+3i, 1), (-(5n+6+3i), 0)}, \dots, (6n-4, 1), (-(6n-3), 0));$$

$$T_9 = ((6n-2, 1), (-6n, 0), (6n, 1), (6n-1, 0), (-(6n-2), 1), (6n+1, 0));$$

$$T_{10} = ((-(6n-5), 1), (6n-3, 0), \dots, \underline{(-(6n-5-3i), 1), (6n-3-3i, 0)}, \dots, (-(5n+4), 1), (5n+6, 0));$$

$$T_{11} = ((-(5n+1), 1), (5n+1, 0));$$

$$T_{12} = ((-(5n+5), 1), (5n+4, 0), \dots, \underline{(-(5n+5+3i), 1), (5n+4+3i, 0)}, \dots, (-(6n-4), 1), (6n-5, 0), (-(6n-1), 1)).$$

Case 2: $k \equiv 5 \pmod{12}$, $k \geq 29$.

The cycle C_6 is the concatenation of the sequences T_1, T_2, \dots, T_{12} , where

$$T_1 = ((-(6n+1), 0), (-(6n-1), 0));$$

$$T_2 = ((6n-6, 1), (-(6n-5), 0), \dots, \underline{(6n-6-3i, 1), (-(6n-5-3i), 0)}, \dots, (5n+4, 1), (-(5n+5), 0));$$

$$T_3 = ((5n+1, 1), (-(5n+1), 0));$$

$$T_4 = ((5n+6, 1), (-(5n+4), 0), \dots, \underline{(5n+6+3i, 1), (-(5n+4+3i), 0)}, \dots, (6n-4, 1), (-(6n-6), 0));$$

$$T_5 = ((6n-2, 1), (-6n, 0), (6n, 1), (6n-1, 0), (-(6n-3), 1), (6n-3, 0), (-(6n-2), 1), (6n+1, 0));$$

$$T_6 = ((-(6n-4), 1), (6n-5, 0), \dots, \underline{(-(6n-4-3i), 1), (6n-5-3i, 0)}, \dots, (-(5n+6), 1), (5n+5, 0));$$

$$T_7 = ((-(5n+3), 1), (5n+2, 0), (-(5n+2), 1), (-(5n+2), 0), (5n+2, 1), (-(5n+3), 0));$$

$$T_8 = ((5n+5, 1), (-(5n+6), 0), \dots, \underline{(5n+5+3i, 1), (-(5n+6+3i), 0)}, \dots, (6n-5, 1), (-(6n-4), 0));$$

$$T_9 = ((6n+1, 1), (-(6n-2), 0), (6n-3, 1), (-(6n-3), 0), (6n-1, 1), (6n, 0), (-6n, 1), (6n-2, 0));$$

$$T_{10} = ((-(6n-6), 1), (6n-4, 0), \dots, \underline{(-(6n-6-3i), 1), (6n-4-3i, 0)}, \dots, (-(5n+4), 1), (5n+6, 0));$$

$$T_{11} = ((-(5n+1), 1), (5n+1, 0));$$

$$T_{12} = ((-(5n+5), 1), (5n+4, 0), \dots, \underline{(-(5n+5+3i), 1), (5n+4+3i, 0)}, \dots, (-(6n-5), 1), (6n-6, 0), (-(6n-1), 1)).$$

Case 3: $k \equiv 9 \pmod{12}$, $k \geq 21$.

The cycle C_6 is the concatenation of the sequences $T_1, T_2, \dots, T_4, (6n+1, 0), T_5, T_6, T_7, (6n+1, 1), T_8, T_9, \dots, T_{11}$ defined as follows:

$$\begin{aligned}
T_1 &= ((-(6n+1), 0), (-(6n-1), 0), (6n-1, 1), (6n, 0), (-(6n, 1))); \\
T_2 &= ((6n-4, 0), (-(6n-3), 1), \dots, \underline{(6n-4-3i, 0), (-(6n-3-3i), 1)}, \dots, (5n+4, 0), (-(5n+5), 1)); \\
T_3 &= ((5n+1, 0), (-(5n+1), 1)); \\
T_4 &= ((5n+6, 0), (-(5n+4), 1), \dots, \underline{(5n+6+3i, 0), (-(5n+4+3i), 1)}, \dots, (6n-2, 0), (-(6n-4), 1)); \\
T_5 &= ((-(6n-2), 1), (6n-3, 0), \dots, \underline{(-(6n-2-3i), 1), (6n-3-3i, 0)}, \dots, (-(5n+6), 1), (5n+5, 0)); \\
T_6 &= ((-(5n+3), 1), (5n+2, 0), (-(5n+2), 1), (-(5n+2), 0), (5n+2, 1), (-(5n+3), 0)); \\
T_7 &= ((5n+5, 1), (-(5n+6), 0), \dots, \underline{(5n+5+3i, 1), (-(5n+6+3i), 0)}, \dots, (6n-3, 1), (-(6n-2), 0)); \\
T_8 &= ((-(6n-4), 0), (6n-2, 1), \dots, \underline{(-(6n-4-3i), 0), (6n-2-3i, 1)}, \dots, (-(5n+4), 0), (5n+6, 1)); \\
T_9 &= ((-(5n+1), 0), (5n+1, 1)); \\
T_{10} &= ((-(5n+5), 0), (5n+4, 1), \dots, \underline{(-(5n+5+3i), 0), (5n+4+3i, 1)}, \dots, (-(6n-3), 0), (6n-4, 1)); \\
T_{11} &= ((-6n, 0), (6n, 1), (6n-1, 0), (-(6n-1), 1)).
\end{aligned}$$

We can easily check that F satisfies all the conditions of Lemma 2.5. \square

Lemma 2.7. *For any $k \geq 11$ and $k \equiv 3 \pmod{4}$, there exists a k -ARCS($6k+1$).*

Proof: Let the vertex set be $(Z_{3k} \times Z_2) \cup \{\infty\}$ and $(a, b) = (0, 1)$. Let $k = 4n+3$, $n \geq 2$. The six initial cycles in F are given as below.

We first define the first four cycles C_i , $1 \leq i \leq 4$, where

$$\begin{aligned}
C_1 &= ((n+2, 0), (-(n+2), 0), \dots, \underline{(n+2+i, 0), (-(n+2+i), 0)}, \dots, (3n+2, 0), (-(3n+2), 0), (-(n-1), 0)); \\
C_2 &= ((n+2, 1), (-(n+2), 1), \dots, \underline{(n+2+i, 1), (-(n+2+i), 1)}, \dots, (3n+2, 1), (-(3n+2), 1), (-(n-1), 1)); \\
C_3 &= ((5n+3, 0), (-(5n+3), 1), \dots, \underline{(5n+3-i, 0), (-(5n+3-i), 1)}, \dots, (3n+3, 0), (-(3n+3), 1), \\
&\quad (-(5n+4), 0)); \\
C_4 &= ((5n+3, 1), (-(5n+3), 0), \dots, \underline{(5n+3-i, 1), (-(5n+3-i), 0)}, \dots, (3n+3, 1), (-(3n+3), 0), \\
&\quad (-(5n+4), 1)).
\end{aligned}$$

To construct C_5 and C_6 , we first deal with $k \in \{11, 15, 19\}$:

$$\begin{aligned}
k = 11: \quad & C_5 = (\infty, (-2, 1), (2, 1), (1, 1), (3, 1), (-3, 0), (-3, 1), (3, 0), (1, 0), (-2, 0), (2, 0)); \\
& C_6 = ((16, 0), (-16, 0), (15, 1), (-15, 0), (14, 1), (15, 0), (-15, 1), (14, 0), (16, 1), (0, 0), (-16, 1)); \\
k = 15: \quad & C_5 = (\infty, (2, 1), (3, 1), (-3, 1), (1, 1), (-1, 1), (4, 1), (-4, 0), (-4, 1), (4, 0), (1, 0), (-1, 0), (3, 0), \\
&\quad (-3, 0), (2, 0)); \\
& C_6 = ((22, 0), (-22, 0), (21, 1), (-21, 0), (20, 1), (-20, 0), (19, 1), (20, 0), (-20, 1), (19, 0), (-22, 1), \\
&\quad (0, 0), (22, 1), (21, 0), (-21, 1)); \\
k = 19: \quad & C_5 = (\infty, (-2, 1), (2, 1), (1, 1), (3, 1), (-4, 1), (4, 1), (-1, 1), (5, 1), (-5, 0), (-5, 1), (5, 0), (-2, 0), \\
&\quad (1, 0), (3, 0), (-1, 0), (4, 0), (-4, 0), (2, 0)); \\
& C_6 = ((28, 0), (-28, 0), (27, 1), (-27, 0), (26, 1), (-26, 0), (25, 1), (-25, 0), (24, 1), (25, 0), (-25, 1), \\
&\quad (24, 0), (-27, 1), (27, 0), (-26, 1), (26, 0), (28, 1), (0, 0), (-28, 1)).
\end{aligned}$$

For $k > 19$, we first consider the cycle C_5 . It is the concatenation of the sequences ∞ , T_1, T_2, \dots, T_5 , there are the following three cases.

Case 1: $k \equiv 3 \pmod{12}$, $k \geq 27$.

$$\begin{aligned}
T_1 &= ((2, 1), (3, 1), (-3, 1), (1, 1), (-1, 1), (4, 1)); \\
T_2 &= ((-4, 1), (6, 1), (-6, 1), (5, 1), (-2, 1), (7, 1), \dots, \underline{(-(4+3i), 1), (6+3i, 1), (-(6+3i), 1), (5+3i, 1)}, \\
&\quad \underline{(-(2+3i), 1), (7+3i, 1)}, \dots, (-(n-2), 1), (n, 1), (-n, 1), (n-1, 1), (-(n-4), 1), (n+1, 1));
\end{aligned}$$

$$T_3 = ((-(n+1), 0), (-(n+1), 1));$$

$$T_4 = ((n+1, 0), (-(n-4), 0), (n-1, 0), (-n, 0), (n, 0), (-(n-2), 0), \dots, \underline{(n+1-3i, 0), (-(n-4-3i), 0)}, \underline{(n-1-3i, 0), (-(n-3i), 0), (n-3i, 0), (-(n-2-3i), 0)}, \dots, (7, 0), (-2, 0), (5, 0), (-6, 0), (6, 0), (-4, 0), (4, 0));$$

$$T_5 = ((1, 0), (-1, 0), (3, 0), (-3, 0), (2, 0)).$$

Case 2: $k \equiv 7 \pmod{12}$, $k \geq 31$.

$$T_1 = ((-2, 1), (2, 1), (1, 1), (3, 1), (-4, 1), (4, 1), (-1, 1), (5, 1));$$

$$T_2 = ((-7, 1), (7, 1), (-3, 1), (6, 1), (-5, 1), (8, 1), \dots, \underline{(-7+3i, 1), (7+3i, 1), (-(3+3i), 1), (6+3i, 1)}, \underline{(-(5+3i), 1), (8+3i, 1)}, \dots, (-n, 1), (n, 1), (-(n-4), 1), (n-1, 1), (-(n-2), 1), (n+1, 1));$$

$$T_3 = ((-(n+1), 0), (-(n+1), 1));$$

$$T_4 = ((n+1, 0), (-(n-2), 0), (n-1, 0), (-(n-4), 0), (n, 0), (-n, 0), \dots, \underline{(n+1-3i, 0), (-(n-2-3i), 0)}, \underline{(n-1-3i, 0), (-(n-4-3i), 0), (n-3i, 0), (-(n-3i), 0)}, \dots, (8, 0), (-5, 0), (6, 0), (-3, 0), (7, 0), (-7, 0), (5, 0));$$

$$T_5 = ((-1, 0), (4, 0), (-4, 0), (3, 0), (1, 0), (-2, 0), (2, 0)).$$

Case 3: $k \equiv 11 \pmod{12}$, $k \geq 23$.

$$T_1 = ((-2, 1), (2, 1), (1, 1), (3, 1));$$

$$T_2 = ((-5, 1), (5, 1), (-1, 1), (4, 1), (-3, 1), (6, 1), \dots, \underline{(-(5+3i), 1), (5+3i, 1), (-(1+3i), 1), (4+3i, 1)}, \underline{(-(3+3i), 1), (6+3i, 1)}, \dots, (-n, 1), (n, 1), (-(n-4), 1), (n-1, 1), (-(n-2), 1), (n+1, 1));$$

$$T_3 = ((-(n+1), 0), (-(n+1), 1));$$

$$T_4 = ((n+1, 0), (-(n-2), 0), (n-1, 0), (-(n-4), 0), (n, 0), (-n, 0), \dots, \underline{(n+1-3i, 0), (-(n-2-3i), 0)}, \underline{(n-1-3i, 0), (-(n-4-3i), 0), (n-3i, 0), (-(n-3i), 0)}, \dots, (6, 0), (-3, 0), (4, 0), (-1, 0), (5, 0), (-5, 0));$$

$$T_5 = ((3, 0), (1, 0), (-2, 0), (2, 0)).$$

C_6 is the concatenation of the sequences T_1, T_2, \dots, T_5 . We distinguish two cases.

Case 1: $k \equiv 3 \pmod{8}$, $k \geq 27$.

$$T_1 = ((6n+4, 0), (-(6n+4), 0));$$

$$T_2 = ((6n+3, 1), (-(6n+3), 0), \dots, \underline{(6n+3-i, 1), (-(6n+3-i), 0)}, \dots, (5n+5, 1), (-(5n+5), 0));$$

$$T_3 = ((5n+4, 1), (5n+5, 0), (-(5n+5), 1), (5n+4, 0));$$

$$T_4 = ((-(5n+7), 1), (5n+7, 0), (-(5n+6), 1), (5n+6, 0), \dots, \underline{(-(5n+7+2i), 1), (5n+7+2i, 0)}, \underline{(-(5n+6+2i), 1), (5n+6+2i, 0)}, \dots, (-(6n+3), 1), (6n+3, 0), (-(6n+2), 1), (6n+2, 0));$$

$$T_5 = ((6n+4, 1), (0, 0), (-(6n+4), 1)).$$

Case 2: $k \equiv 7 \pmod{8}$, $k \geq 23$.

$$T_1 = ((6n+4, 0), (-(6n+4), 0));$$

$$T_2 = ((6n+3, 1), (-(6n+3), 0), \dots, \underline{(6n+3-i, 1), (-(6n+3-i), 0)}, \dots, (5n+5, 1), (-(5n+5), 0));$$

$$T_3 = ((5n+4, 1), (5n+5, 0), (-(5n+5), 1), (5n+4, 0));$$

$$T_4 = ((-(5n+7), 1), (5n+7, 0), (-(5n+6), 1), (5n+6, 0), \dots, \underline{(-(5n+7+2i), 1), (5n+7+2i, 0)},$$

$(-(5n+6+2i), 1), (5n+6+2i, 0), \dots, (-(6n+2), 1), (6n+2, 0), (-(6n+1), 1), (6n+1, 0));$
 $T_5 = ((-(6n+4), 1), (0, 0), (6n+4, 1), (6n+3, 0), (-(6n+3), 1)).$

F satisfies all the conditions of Lemma 2.5 and the assertion holds. \square

2.3 Constructions for k -ARCS($2kt+1$)

For the main recursive constructions, we need the definition of cycle frames. A 2-regular subgraph of a complete multipartite graph covering all vertices except those belonging to one part G is said to be a *holey 2-factor* missing G . We will also frequently speak of a *holey C_k -factor* to mean a holey 2-factor whose cycles have length k .

Let H be a graph $K_u[g]$ with u parts G_1, G_2, \dots, G_u . A partition of $E(H)$ into holey 2-factors of $H - G_i$ ($1 \leq i \leq u$) is said to be a *cycle frame of type g^u* . Further, if all holey 2-factors of a cycle frame of type g^u are C_k -factors, then we denote the cycle frame by $(k, 1)$ -cycle frame of type g^u . We write it as $(k, 1)$ -CF(g^u) for brevity. Many authors have contributed to prove the following results.

Theorem 2.8. ([7, 9, 11, 17, 18, 30, 35, 40]) *There exists a $(k, 1)$ -CF(g^u) if and only if $g \equiv 0 \pmod{2}$, $g(u-1) \equiv 0 \pmod{k}$, $u \geq 3$ when k is even, $u \geq 4$ when k is odd, except a $(6, 1)$ -CF(6^3).*

H. Cao et al. [11] give the following general recursive construction for an almost resolvable cycle system of order n by using cycle frames.

Construction 2.9. ([11]) *Suppose there exists a $(k, 1)$ -CF($2k$) ^{t} and a k -ARCS($2k+1$). Then there exists a k -ARCS($2kt+1$).*

Theorem 2.10. *For any odd $k \geq 11$, there exists a k -ARCS($2kt+1$), where $t \geq 1$ and $t \neq 2$.*

Proof: When $t = 1, 3$, the conclusion comes from Lemmas 2.4, 2.6 and 2.7.

When $t \geq 4$, for any odd $k \geq 5$, there exists a $(k, 1)$ -CF($2k$) ^{t} by Theorem 2.8. Applying Construction 2.9 with a k -ARCS($2k+1$) which exists by Lemma 2.4, we can obtain the required k -ARCS($2kt+1$). \square

Further, we use the ARCSs as a tool to obtain some results of the Hamilton-Waterloo problem. First, we give some recursive constructions.

3 Some new constructions

Before giving their constructions, we still need some definitions in graph theory. For more general concepts of graph theory, see [38].

Given a graph G , $G[n]$ is the lexicographic product of G with the empty graph on n points. Specifically, the point set is $\{x_i : x \in V(G), i \in Z_n\}$ and $(x_i, y_j) \in E(G[n])$ if and only if $(x, y) \in E(G), i, j \in Z_n$. In the following we will denote by $C_m[n]$ the lexicographic product of C_m with the empty graph on n points.

We need the following known results for our recursive constructions.

Theorem 3.1. ([11, 33]) *There exists a C_m -factorization of $C_m[n]$ for $m \geq 3$ and $n \geq 1$ except only for $(m, n) = (3, 6)$ and $(m, n) \in \{(l, 2) \mid l \geq 3 \text{ is odd}\}$.*

Theorem 3.2. ([25]) *There exists a C_{mn} -factorization of $C_m[n]$ for $m \geq 3$ and $n \geq 1$.*

For the next recursive construction, we need more notations. When $g(u-1) \equiv 1 \pmod{2}$, it is easy to see that an $\text{HW}(K_u[g]; m, n; \alpha, \beta)$ can not exist. In this case, by simple computation, we know that it is possible to partition $E(K_u[g])$ into a 1-factor, α C_m -factors and β C_n -factors where $\alpha + \beta = \lfloor \frac{g(u-1)}{2} \rfloor$. For brevity, we still use $\text{HW}(K_u[g]; m, n; \alpha, \beta)$ to denote such a decomposition. Further, denote by $\text{HWP}(K_u[g]; m, n)$ the set of (α, β) for which a solution $\text{HW}(K_u[g]; m, n; \alpha, \beta)$ exists. Similarly, $\text{HWP}(C_u[g]; m, n)$ denote the set of (α, β) for which a solution $\text{HW}(C_u[g]; m, n; \alpha, \beta)$ exists.

Construction 3.3. ([39]) *If there exist an $\text{HW}(K_u[g]; m, n; \alpha, \beta)$ and an $\text{HW}(g; m, n; \alpha', \beta')$, then an $\text{HW}(gu; m, n; \alpha + \alpha', \beta + \beta')$ exists.*

Construction 3.4. ([39]) *If $(\alpha, \beta) \in \text{HWP}(K_u[g]; m, n)$, $(t_i, s - t_i) \in \text{HWP}(C_m[s]; m', n')$, $1 \leq i \leq \alpha$, and $(r_j, s - r_j) \in \text{HWP}(C_n[s]; m', n')$, $1 \leq j \leq \beta$, then $(\alpha', \beta') \in \text{HWP}(K_u[gs]; m', n')$, where $\alpha' = \sum_{i=1}^{\alpha} t_i + \sum_{j=1}^{\beta} r_j$ and $\beta' = (\alpha + \beta)s - \alpha'$.*

Theorem 3.5. ([8]) *If m and n are odd integers with $n \geq m \geq 3$, $0 \leq \alpha \leq n$, then $(\alpha, \beta) \in \text{HWP}(C_m[n]; m, n)$, if and only if $\beta = n - \alpha$, except possibly when $\alpha = 2, 4$, $\beta = 1, 3$, or $(m, n, \alpha) = (3, 11, 6), (3, 13, 8), (3, 15, 8)$.*

Let Γ be a finite additive group and let S be a subset of $\Gamma \setminus \{0\}$ such that the opposite of every element of S also belongs to S . The Cayley graph over Γ with connection set S , denoted by $\text{Cay}(\Gamma, S)$, is the graph with vertex set Γ and edge set $E(\text{Cay}(\Gamma, S)) = \{(a, b) \mid a, b \in \Gamma, a - b \in S\}$. It is quite obvious that $\text{Cay}(\Gamma, S) = \text{Cay}(\Gamma, \pm S)$.

Theorem 3.6. ([39]) *Let $n \geq 3$. If $a \in Z_n$, the order of a is greater than 3 and $(i, m) = 1$, then there is a C_m -factorization of $\text{Cay}(Z_m \times Z_n, \{\pm i\} \times (\pm\{0, a, 2a\}))$.*

Theorem 3.7. ([8]) *Let $n \geq m \geq 3$ be odd integers and let $0 < d_1, d_2 < n$. If any linear combination of d_1 and d_2 is coprime to n , then there exist four C_n -factors which form a C_n -factorization of $\text{Cay}(Z_m \times Z_n, \{\pm 1\} \times (\pm\{d_1, d_2\}))$.*

Theorem 3.8. ([8]) *Let $n \geq m \geq 3$ be odd integers and let $0 < d < n$ be coprime to n . There exist two C_n -factors which form a C_n -factorization of $\text{Cay}(Z_m \times Z_n, \{\pm 1\} \times \{\pm d\})$.*

Lemma 3.9. $(2, n-2) \in \text{HWP}(C_m[n]; m, n)$ for any odd integers $n \geq m \geq 3$ and $n \equiv 3 \pmod{6} \geq 9$.

Proof: Let the vertex set be $\Gamma = Z_m \times Z_n$ and $n = 3d$. Let $C_j = ((0, 0), (1, b_{j1}), \dots, (m-1, b_{j,m-1}))$, $1 \leq j \leq 2$, where

$$b_{11} = -b_{21} = d, \quad b_{12} = -b_{22} = 2d, \quad b_{jt} = b_{j,(t-2)}, \quad t \geq 3.$$

Then each C_j will generate a C_m -factor by $(-, +1 \pmod{n})$. Thus we get two C_m -factors which form a C_m -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times \{\pm d\})$.

For $n-2$ C_n -factors, five of which can be obtained from a C_n -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times (\pm\{0, 1, 2\}))$ by Theorem 3.6. The other $n-7$ C_n -factors come from a C_n -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times (\pm\{3, 4, \dots, \frac{n-1}{2}\} \setminus \{\pm d\}))$. In fact, when $n \equiv 3 \pmod{12}$, there exist 4 C_n -factors which form a C_n -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times (\pm\{2j-1, 2j\}))$, $2 \leq j \leq \frac{d-1}{2}$ by Theorem 3.7. There exist 4 C_n -factors which form a C_n -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times (\pm\{d+2j-1, d+2j\}))$, $1 \leq j \leq \frac{d-1}{4}$ by the same theorem; when $n \equiv 9 \pmod{12}$, there exist 4 C_n -factors which form a C_n -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times (\pm\{2j-1, 2j\}))$, $2 \leq j \leq \frac{d-1}{2}$ by Theorem 3.7. There exist 4 C_n -factors which form a C_n -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times (\pm\{d+2j-1, d+2j\}))$, $1 \leq j \leq \frac{d-3}{4}$ by the same theorem. Further, there exist 2 C_n -factors which form a C_n -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times \{\pm \frac{n-1}{2}\})$ by Theorem 3.8.

□

Lemma 3.10. $(4, 5) \in \text{HWP}(C_m[9]; m, 9)$ for any odd integer $m \geq 5$.

Proof: Let the vertex set be $\Gamma = Z_m \times Z_9$. Let $C_j = ((0, 0), (1, b_{j1}), \dots, (m-1, b_{j,m-1}))$, $1 \leq j \leq 4$, where $b_{jt} = b_{j,t-2}$, $t \geq 5$ and

$$\begin{aligned} b_{11} = -b_{21} = 3, \quad b_{12} = -b_{22} = 7, \quad b_{13} = -b_{23} = 3, \quad b_{14} = -b_{24} = 6, \\ b_{31} = -b_{41} = 5, \quad b_{32} = -b_{42} = 2, \quad b_{33} = -b_{43} = 8, \quad b_{34} = -b_{44} = 4. \end{aligned}$$

Each C_j will generate a C_m -factor by $(-, +1 \pmod{9})$. Thus we get 4 C_m -factors which form a C_m -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times (\pm\{3, 4\}))$. The required 5 C_9 -factors can be obtained from a C_n -factorization of $\text{Cay}(\Gamma, \{\pm 1\} \times (\pm\{0, 1, 2\}))$ by Theorem 3.6. □

Construction 3.11. *Let $k \geq 3$. If a k -ARCS($2kt+1$) exists, then $\text{Cay}(Z_k \times Z_{2kt+1}, \{0\} \times (Z_{2kt+1} \setminus \{0\}) \cup \{\pm 1\} \times \{0\})$ can be decomposed into $kt+1$ C_k -factors.*

Proof: Let $A_i = \{i\} \times Z_{2kt+1}$, $i \in Z_k$. Put the blocks of a k -ARCS($2kt+1$) on each A_i . For $0 \leq j \leq kt-1$, without loss of generality, suppose the vertex (i, j) doesn't appear in the j th almost parallel class P_{ij} . And $\{(i, j) \mid 0 \leq j \leq kt-1\}$ appear in the half-parallel class Q_i . Let $S_j = (\bigcup_{i \in Z_k} P_{ij}) \cup ((0, j), (1, j), \dots, (k-1, j))$, $0 \leq j \leq kt-1$. It is easy to check that each S_j is a C_k -factor. $T = (\bigcup_{i \in Z_k} Q_i) \cup \{((0, j), (1, j), \dots, (k-1, j)) \mid kt \leq j \leq 2kt\}$ is a C_k -factor. Namely, we obtain $kt+1$ C_k -factors. \square

In order to give the next construction, we first introduce the following lemma.

Lemma 3.12. *$\text{Cay}(Z_{17} \times Z_{35}, \{\pm 1\} \times (Z_{35} \setminus \{0\}))$ can be decomposed into 28 C_{17} -factors and 6 C_{35} -factors.*

Proof: Let $A = [a_{ij}]$ be the 28×17 array, where

$$\begin{array}{llllll} a_{11} = 3, & a_{12} = -15, & a_{13} = 12, & a_{21} = 6, & a_{22} = 12, & a_{23} = -18, \\ a_{31} = -12, & a_{32} = 6, & a_{33} = 6, & a_{41} = 15, & a_{42} = -18, & a_{43} = 3, \\ a_{51} = -18, & a_{52} = 3, & a_{53} = 15, & a_{61} = 2, & a_{62} = 7, & a_{63} = -9, \\ a_{71} = 7, & a_{72} = -9, & a_{73} = 2, & a_{81} = -9, & a_{82} = 2, & a_{83} = 7, \\ a_{91} = 1, & a_{92} = 10, & a_{93} = -11, & a_{10,1} = 10, & a_{10,2} = -11, & a_{10,3} = 1, \\ a_{11,1} = -11, & a_{11,2} = 1, & a_{11,3} = 10, & a_{12,1} = 5, & a_{12,2} = 14, & a_{12,3} = 16, \\ a_{13,1} = 14, & a_{13,2} = 16, & a_{13,3} = 5, & a_{14,1} = 16, & a_{14,2} = 5, & a_{14,3} = 14, \end{array}$$

$$a_{ij} = -a_{i-14,j}, \quad 15 \leq i \leq 28, \quad 1 \leq j \leq 3,$$

$$a_{i4} = -a_{i5} = a_{i1}, a_{ij} = a_{i,j-2}, \quad 1 \leq i \leq 28, \quad 6 \leq j \leq 17.$$

Let $b_{ij} = \sum_{l=1}^j a_{il}$. For $1 \leq i \leq 28$, each $C_i = (0, 1_{b_{i1}}, 2_{b_{i2}}, \dots, 16_{b_{i,16}})$ can get a C_{17} -factor by $(-, +1 \pmod{35})$. It is easy to check that these 28 C_{17} -factors are a C_{17} -factorization of $\text{Cay}(Z_{17} \times Z_{35}, \{\pm 1\} \times (Z_{35} \setminus (\pm\{0, 4, 8, 13\})))$.

For C_{35} -factors, four of which come from a C_{35} -factorization of $\text{Cay}(Z_{17} \times Z_{35}, \{\pm 1\} \times (\pm\{4, 8\}))$ by Theorem 3.7. The other 2 C_{35} -factors come from a C_{35} -factorization of $\text{Cay}(Z_{17} \times Z_{35}, \{\pm 1\} \times (\{\pm 13\}))$ by Theorem 3.8. \square

Construction 3.13. *Let k be an odd integer. Then $\text{Cay}(Z_k \times Z_{2kt+1}, \{\pm 1\} \times (Z_{2kt+1} \setminus \{0\}))$ can be decomposed into $2l$ C_k -factors and $2kt - 2l$ C_{2kt+1} -factors for $l \notin \{1, 2, kt-1, kt\}$.*

Proof: By Lemmas 3.6, 3.11 and 3.13 in [8] we know that $\text{Cay}(Z_k \times Z_{2kt+1}, \{\pm 1\} \times Z_{2kt+1})$ can be decomposed into $2l$ C_k -factors and $2kt - 2l + 1$ C_{2kt+1} -factors for any $l \notin \{1, 2, kt-1, kt\}$. In these constructions, we can always find five C_{2kt+1} -factors which form a C_{2kt+1} -factorization of $\text{Cay}(Z_k \times Z_{2kt+1}, \{\pm 1\} \times (\pm\{0, d, 2d\}))$ such that $(d, 2kt+1) = 1$ except for $(k, t, l) = (17, 1, 14)$. By Theorem 3.7 we know that $\text{Cay}(Z_k \times Z_{2kt+1}, \{\pm 1\} \times (\pm\{d, 2d\}))$

can be partitioned into 4 C_{2kt+1} -factors. Thus $\text{Cay}(Z_k \times Z_{2kt+1}, \{\pm 1\} \times (Z_{2kt+1} \setminus \{0\}))$ with $(k, t, l) \neq (17, 1, 14)$ can be decomposed into $2l$ C_k -factors and $2kt - 2l$ C_{2kt+1} -factors for $l \notin \{1, 2, kt - 1, kt\}$. Further, $\text{Cay}(Z_{17} \times Z_{35}, \{\pm 1\} \times (Z_{35} \setminus \{0\}))$ can be decomposed into 28 C_{17} -factors and 6 C_{35} -factors by Lemma 3.12. \square

Construction 3.14. *If a k -CF(2^u) exists, then $\text{Cay}(Z_k \times Z_{2u}, \{0\} \times (Z_{2u} \setminus \{0\}) \cup \{\pm 1\} \times \{0\})$ can be decomposed into u C_k -factors and a 1-factor.*

Proof: Let $A_i = \{i\} \times Z_{2u}$, $i \in Z_k$. The 1-factor is $\{(i, 2j), (i, 2j + 1) \mid i \in Z_k, 0 \leq j \leq u - 1\}$. For each A_i , put the blocks of a k -CF(2^u) on the vertex set A_i , the parts of the cycle frame are $G_{ij} = \{(i, 2j), (i, 2j + 1)\}$, $0 \leq j \leq u - 1$. Denote the holey 2-factor of G_{ij} by P_{ij} . Let $S_j = (\bigcup_{i \in Z_k} P_{ij}) \cup \{((0, 2j), (1, 2j)), \dots, (k-1, 2j), ((0, 2j+1), (1, 2j+1)), \dots, (k-1, 2j+1)\}$, $0 \leq j \leq u - 1$. It is easy to check that each S_j is a C_k -factor. That is to say, we get u C_k -factors. \square

4 Main results

In order to prove the main results, we need the following six direct constructions.

Lemma 4.1. $(6, 5) \in \text{HWP}(C_3[11]; 3, 11)$.

Proof: Let the vertex set be $(Z_6 \times Z_5) \cup \{a, b, c\}$, and the three parts of $C_3[11]$ be $\{a, \{2, 5\} \times Z_5\}$, $\{b, \{0, 3\} \times Z_5\}$ and $\{c, \{1, 4\} \times Z_5\}$.

For the required 6 C_3 -factors, five of which will be generated from an initial C_3 -factor P by $(-, +1 \pmod{5})$. The last C_3 -factor is $\{(a, b, c), (0_i, 1_{2+i}, 2_{4+i}), (3_{3+i}, 4_{2+i}, 5_{3+i}) \mid i \in Z_5\}$. 5 C_{11} -factors will be generated from an initial C_{11} -factor Q by $(-, +1 \pmod{5})$. The cycles of P and Q are listed below.

$$\begin{array}{llllll} P & (a, 0_0, 4_4) & (b, 2_3, 1_0) & (c, 2_4, 3_1) & (1_1, 3_3, 5_0) & (2_2, 4_0, 0_2) & (0_1, 5_2, 1_4) \\ & (1_2, 2_1, 3_2) & (3_4, 2_0, 4_2) & (5_1, 0_3, 4_3) & (1_3, 0_4, 5_4) & (3_0, 4_1, 5_3) & \\ Q & (a, 1_1, 5_1, 3_0, 1_4, 0_4, 4_1, 2_2, 3_2, 1_3, 3_1) & (b, 5_0, 0_1, 1_2, 2_3, 4_3, 0_2, 2_4, 0_3, 2_1, 4_2) & & & & \\ & (c, 0_0, 5_2, 1_0, 2_0, 3_3, 4_0, 5_4, 4_4, 3_4, 5_3) & & & & & \end{array}$$

\square

Lemma 4.2. $(8, 5) \in \text{HWP}(C_3[13]; 3, 13)$.

Proof: Let the vertex set be $\Gamma = Z_{13} \times Z_3$, and the three parts of $C_3[13]$ be $Z_{13} \times \{i\}$, $i \in Z_3$.

For C_3 -factors, five of which come from a C_3 -factorization of $\text{Cay}(\Gamma, \pm\{0, 1, 2\} \times \{\pm 1\})$ by Lemma 3.6. The 3-cycle $(0_0, 4_1, 7_2)$ can generate a C_3 -factor F by $(+1 \pmod{13}, -)$, then the last 3 C_3 -factors can be obtained from F by $(-, +1 \pmod{3})$. 5 C_{13} -factors will be generated from 5 13-cycles in Q by $(-, +1 \pmod{3})$ since the 13 elements in the first coordinate of each 13-cycle are different modular 13. The cycles of Q are listed below.

$$\begin{array}{ll}
Q & (0_0, 5_2, 10_1, 2_2, 7_1, 12_0, 4_1, 9_0, 11, 6_0, 11_2, 3_0, 8_2) \quad (0_0, 7_1, 1_2, 4_1, 11_2, 3_1, 6_0, 10_2, 2_1, 8_0, 12_2, 9_0, 5_1) \\
& (0_0, 10_1, 3_0, 7_2, 12_0, 2_2, 9_0, 1_2, 6_0, 11_1, 5_2, 8_1, 4_2) \quad (0_0, 9_1, 3_2, 8_0, 11_2, 1_1, 10_2, 4_0, 7_2, 2_1, 5_0, 12_1, 6_2) \\
& (0_0, 3_2, 12_0, 4_2, 9_0, 6_1, 2_2, 11_0, 7_1, 10_0, 5_2, 1_0, 8_1)
\end{array}$$

□

Lemma 4.3. $(8, 7) \in \text{HWP}(C_3[15]; 3, 15)$.

Proof: Let the vertex set be Z_{45} , and the three parts of $C_3[15]$ be $\{3i + j \mid i \in Z_{15}, j \in Z_3\}$. Since all 3 elements of each cycle in P are different modular 3, so all 8 C_3 -factors will be generated from 8 3-cycles in P by $+3 \pmod{45}$. And the required 7 C_{15} -factors will be generated from 7 15-cycles in Q by $+15 \pmod{45}$. The cycles of P and Q are listed below.

$$\begin{array}{llllllll}
P & (0, 1, 2) & (0, 4, 8) & (0, 5, 7) & (0, 10, 17) & (0, 11, 16) & (0, 13, 23) & (0, 14, 22) & (0, 19, 32) \\
Q & (0, 34, 6, 41, 3, 43, 9, 44, 10, 27, 2, 16, 5, 22, 38) & (0, 20, 1, 3, 2, 19, 21, 14, 43, 26, 40, 9, 38, 42, 37) \\
& (0, 28, 2, 12, 1, 18, 7, 21, 5, 9, 8, 19, 44, 25, 41) & (0, 29, 1, 17, 28, 3, 23, 4, 20, 39, 22, 42, 26, 6, 40) \\
& (0, 31, 6, 32, 3, 29, 9, 34, 11, 37, 23, 42, 40, 20, 43) & (0, 25, 2, 9, 1, 6, 5, 19, 33, 23, 43, 12, 11, 22, 44) \\
& (0, 26, 1, 23, 40, 3, 44, 13, 21, 17, 37, 39, 34, 42, 35)
\end{array}$$

□

Combining Theorem 3.5, Lemmas 4.1-4.3, we have the following conclusion.

Lemma 4.4. *If n, m and α are odd integers with $n \geq m \geq 3$, $0 \leq \alpha \leq n$, then $(\alpha, \beta) \in \text{HWP}(C_m[n]; m, n)$, if and only if $\beta = n - \alpha$, except possibly when $\alpha = 2, 4$, $\beta = 1, 3$.*

Lemma 4.5. $(6, 10) \in \text{HWP}(C_3; 3, 11)$.

Proof: Let the vertex set be $(Z_6 \times Z_5) \cup \{a, b, c\}$. For the required 6 C_3 -factors, five of which will be generated from an initial C_3 -factor P by $(-, +1 \pmod{5})$. The last C_3 -factor is $\{(a, b, c), ((3j)_i, (3j+1)_{3+i}, (3j+2)_i) \mid i \in Z_5, j = 0, 1\}$. 10 C_{11} -factors will be generated from an initial C_{11} -factor Q by $(+3 \pmod{6}, +1 \pmod{5})$. P and Q are listed below.

$$\begin{array}{llllll}
P & (a, 1_1, 4_1) & (b, 2_2, 5_2) & (c, 0_0, 3_0) & (3_3, 5_0, 1_4) & (4_4, 0_3, 2_0) & (0_1, 2_4, 3_2) \\
& (1_2, 2_3, 2_1) & (3_4, 4_0, 4_3) & (5_1, 4_2, 5_3) & (0_2, 3_1, 5_4) & (1_3, 0_4, 1_0) \\
Q & (a, 2_3, 1_3, 0_3, 5_1, 4_3, 3_1, 2_0, 3_4, 3_0, 3_2) & (b, 0_2, 2_1, 5_4, 0_4, 4_1, 4_2, 1_0, 5_2, 5_3, 1_4) \\
& (c, 1_1, 4_0, 0_0, 3_3, 1_2, 5_0, 2_4, 4_4, 0_1, 2_2)
\end{array}$$

□

Lemma 4.6. $(9, 8) \in \text{HWP}(C_5; 5, 7)$.

Proof: Let the vertex set be $\Gamma = Z_5 \times Z_7$. For the required 9 C_5 -factors, seven of which will be generated an initial C_5 -factor P by $(-, +1 \pmod{7})$. The last 2 C_5 -factors will be obtained from two cycles $\{(0_0, 4_2, 2_2, 1_4, 3_4), (0_0, 1_4, 3_3, 2_3, 4_1)\}$ by $(-, +1 \pmod{7})$ since the 5 elements in the first coordinate of each cycle are different modular 5. For C_7 -factors, seven of which will be generated an initial C_7 -factor Q by $(-, +1 \pmod{7})$. The last C_7 -factor is $\text{Cay}(\Gamma, \{0\} \times \{\pm 2\})$. The cycles of P and Q are listed below.

$$\begin{array}{llll}
P & (0_0, 1_1, 2_2, 3_3, 4_4) & (0_5, 2_0, 4_2, 1_6, 3_1) & (0_3, 1_2, 2_5, 0_1, 2_1) & (1_4, 2_3, 3_6, 4_5, 0_6) \\
& (4_0, 3_4, 0_4, 3_5, 2_6) & (1_0, 4_1, 2_4, 3_2, 4_6) & (4_3, 1_5, 0_2, 3_0, 1_3) \\
Q & (0_0, 0_3, 2_2, 2_5, 1_1, 1_4, 3_2) & (3_3, 3_6, 1_5, 0_5, 2_1, 4_4, 1_0) & (1_6, 2_6, 2_0, 0_6, 4_6, 4_5, 4_1) \\
& (3_1, 4_3, 2_4, 1_2, 1_3, 0_1, 0_2) & (4_2, 3_4, 3_5, 2_3, 3_0, 4_0, 0_4)
\end{array}$$

□

Lemma 4.7. $(8, 11) \in \text{HWP}(39; 3, 13)$.

Proof: Let the vertex set be $\Gamma = Z_{13} \times Z_3$. For C_3 -factors, five of which will be generated from a C_3 -factorization of $\text{Cay}(\Gamma, \pm\{0, 1, 2\} \times \{\pm 1\})$ by Lemma 3.6. The 3-cycle $(0_0, 4_1, 7_2)$ can generate a C_3 -factor F by $(+1 \pmod{13}, -)$, then the last 3 C_3 -factors can be obtained from F by $(-, +1 \pmod{3})$. 11 C_{13} -factors will be generated from 11 13-cycles in Q by $(-, +1 \pmod{3})$. The cycles of Q are listed below.

Q	$(0_0, 4_2, 2_2, 6_1, 1_1, 3_1, 11_0, 7_1, 5_1, 8_0, 12_2, 9_0, 10_0)$	$(0_0, 5_0, 3_0, 7_2, 2_2, 10_2, 4_0, 1_1, 6_2, 11_0, 12_0, 9_0, 8_0)$
	$(0_0, 9_1, 2_1, 5_0, 4_0, 8_0, 6_0, 10_0, 1_2, 11_2, 7_2, 12_2, 3_2)$	$(0_0, 3_0, 6_0, 1_1, 4_1, 9_2, 5_2, 10_2, 12_2, 7_1, 8_1, 2_2, 11_0)$
	$(0_0, 5_2, 10_1, 1_1, 9_1, 3_1, 6_0, 11_2, 8_0, 2_0, 7_1, 12_0, 4_0)$	$(0_0, 7_1, 10_1, 8_1, 11_1, 3_0, 4_0, 9_0, 1_1, 2_1, 5_1, 12_1, 6_2)$
	$(0_0, 10_1, 2_0, 3_0, 11_0, 5_1, 8_1, 4_2, 12_1, 6_1, 9_1, 1_0, 7_0)$	$(0_0, 12_0, 2_0, 9_1, 4_2, 11_0, 10_0, 6_1, 7_1, 3_1, 8_2, 1_1, 5_1)$
	$(0_0, 2_0, 6_0, 5_0, 10_1, 7_2, 4_0, 12_1, 8_1, 3_2, 9_1, 11_1, 1_0)$	$(0_0, 8_2, 1_2, 12_2, 3_1, 10_1, 2_2, 7_1, 4_1, 6_1, 11_1, 5_1, 9_0)$
	$(0_0, 6_0, 9_2, 7_2, 1_0, 5_2, 12_0, 2_2, 11_2, 4_2, 10_2, 3_1, 8_1)$	

□

Lemma 4.8. $(\alpha, \beta) \in \text{HWP}(9u; u, 9)$ for $u = 5, 7$, $\beta = 9, 11$ and $\alpha + \beta = \frac{9u-1}{2}$.

Proof: There exist an $\text{HW}(K_u[1]; u, 9; \frac{u-1}{2}, 0)$ (from Theorem 1.1), an $\text{HW}(C_u[9]; u, 9; 9, 0)$ (from Theorem 3.5) and an $\text{HW}(C_u[9]; u, 9; \alpha, 9 - \alpha)$ for $\alpha = 2, 4$ (from Lemmas 3.9 and 3.10). Apply Construction 3.4 with $s = 9$ and $t_i \in \{9, \alpha\}$ to obtain an $\text{HW}(K_u[9]; u, 9; \frac{9(u-3)}{2} + \alpha, 9 - \alpha)$. Applying Construction 3.3 with an $\text{HW}(9; u, 9; 0, 4)$ from Theorem 1.1, we can get an $\text{HW}(9u; u, 9; \frac{9(u-3)}{2} + \alpha, 13 - \alpha)$ for $\alpha = 2, 4$. □

Lemma 4.9. $(\frac{39t-11}{2}, 5) \in \text{HWP}(39t; 3, 13)$ for any odd $t > 1$.

Proof: Let $\alpha = \frac{3(t-1)}{2}$. Start with an $\text{HW}(K_t[3]; 3, 13; \alpha, 0)$ from Theorem 1.1, an $\text{HW}(C_3[13]; 3, 13; 13, 0)$ from Theorem 3.5 and an $\text{HW}(C_3[13]; 3, 13; 8, 5)$ from Lemma 4.2. Then apply Construction 3.4 with $s = 13$ and $t_i \in \{8, 13\}$ to get an $\text{HW}(K_t[39]; 3, 13; 13\alpha - 5, 5)$. Apply Construction 3.3 with an $\text{HW}(39; 3, 13; 19, 0)$ from Theorem 1.1 to get the conclusion. □

Lemma 4.10. $(\alpha, \beta) \in \text{HWP}(9tu; u, 9)$ for any odd $t > 1$, $u = \beta = 5, 7$ and $\alpha + \beta = \frac{9tu-1}{2}$.

Proof: There exist an $\text{HW}(K_t[u]; u, 9; \alpha', 0)$, $\alpha' = \frac{u(t-1)}{2}$ (from Theorem 1.1), an $\text{HW}(C_u[9]; u, 9; 9, 0)$ (from Theorem 3.5) and an $\text{HW}(C_u[9]; u, 9; \alpha, 9 - \alpha)$, $\alpha = 2, 4$ (from Lemmas 3.9 and 3.10). Apply Construction 3.4 with $s = 9$ and $t_i \in \{9, \alpha\}$ to get an $\text{HW}(K_t[9u]; u, 9; 9(\alpha' - 1) + \alpha, 9 - \alpha)$. Applying Construction 3.3 with an $\text{HW}(9u; u, 9; \frac{9u-1}{2}, 0)$ from Theorem 1.1, the assertion follows. □

Proof of Theorem 1.3: Combining Theorem 1.2 and Lemmas 4.8-4.10, we can get the conclusion. □

Proof of Theorem 1.4: By Theorem 1.1 the complete graph K_k with vertex set Z_k can be decomposed into $r = \frac{k-1}{2}$ cycles B_1, B_2, \dots, B_r . Without loss of generality, we may suppose that $B_r = (0, 1, 2, \dots, k-1)$. Give each vertex weight $2kt+1$. Let $\Gamma = Z_k \times Z_{2kt+1}$.

When $k=3$ and $t \geq 3$, we have $r=1$. For the cycle B_r , $\text{Cay}(\Gamma, (\{0\} \times (Z_{6t+1} \setminus \{0\})) \cup (\{\pm 1\} \times \{0\}))$ can be decomposed into $3t+1$ C_3 -factors by Construction 3.11, where a 3-ARCS($6t+1$) for $t \geq 3$ exists by Theorem 2.1. By Construction 3.13 $\text{Cay}(\Gamma, \{\pm 1\} \times (Z_{6t+1} \setminus \{0\}))$ can be decomposed into $2l$ C_3 -factors and $6t-2l$ C_{6t+1} -factors, $2l \in \{0, 6, 8, 10, \dots, 6t-4\}$. There are altogether $\alpha = 2l+3t+1$ C_3 -factors and $\beta = 6t-2l$ C_{6t+1} -factors. It is obvious that $\beta \in [4, 5 \dots, 6t-6] \cup \{6t\}$ and β is even.

Next, we consider the case $k \geq 5$. Then we have $r \geq 2$. For each cycle B_i , $1 \leq i \leq r-1$, the graph $C_k[2kt+1]$ can be decomposed into $2kt+1-\beta_i$ C_k -factors and β_i C_{2kt+1} -factors by Theorem 3.5, where $0 \leq \beta_i \leq 2kt+1$, $\beta_i \notin \{1, 3, 2kt-3, 2kt-1\}$. For the cycle B_r , $\text{Cay}(\Gamma, (\{0\} \times (Z_{2kt+1} \setminus \{0\})) \cup (\{\pm 1\} \times \{0\}))$ can be decomposed into $kt+1$ C_k -factors by Construction 3.11, where a k -ARCS($2kt+1$) for $t \geq 1$ ($t \neq 2$ when $k \geq 11$) exists by Theorem 2.1 and Lemma 2.10. By Construction 3.13 $\text{Cay}(\Gamma, \{\pm 1\} \times (Z_{2kt+1} \setminus \{0\}))$ can be decomposed into $2l$ C_k -factors and $2kt-2l$ C_{2kt+1} -factors, $0 \leq 2l \leq 2kt+1$, $l \notin \{1, 2, kt-1, kt\}$. Here we have obtained $2l+kt+1$ C_k -factors and $2kt-2l$ C_{2kt+1} -factors. Finally, we get $\alpha = (2l+kt+1) + \sum_{i=1}^{r-1} (2kt+1-\beta_i)$ C_k -factors and $\beta = (2kt-2l) + \sum_{i=1}^{r-1} \beta_i$ C_{2kt+1} -factors, where $0 \leq \beta_i, 2l \leq 2kt+1$, $\beta_i \notin \{1, 3, 2kt-3, 2kt-1\}$ and $l \notin \{1, 2, kt-1, kt\}$. When $k=5$, then $r=2$ and $\beta = (10t-2l) + \beta_1$, where $2l \in \{0, 6, 8, 10, \dots, 10t-4\}$ and $\beta_1 \in [0, 1, \dots, 10t+1] \setminus \{1, 3, 10t-3, 10t-1\}$. It is easy to check that $\beta \in J \setminus \{20t-3, 20t-1\}$. When $k \geq 7$, then $\beta = (2kt-2l) + \sum_{i=1}^{r-1} \beta_i$, where $2l \in \{0, 6, 8, 10, \dots, 2kt-4\}$ and $\beta_i \in [0, 1, \dots, 2kt+1] \setminus \{1, 3, 2kt-3, 2kt-1\}$. We can check that $\beta \in J$. The proof is complete. \square

Proof of Theorem 1.5: Let $v = 4ktu$, $u \geq 1$. For $u=1$, the conclusion comes from Theorem 1.6 in [27]. For $u \geq 2$, start with an $\text{HW}(K_u[4k]; 4k, 4kt; 2k(u-1), 0)$, an $\text{HW}(C_{4k}[t]; 4k, 4kt; t, 0)$ and an $\text{HW}(C_{4k}[t]; 4k, 4kt; 0, t)$ from Theorems 1.1, 3.1 and 3.2, respectively. And apply Construction 3.4 with $s=t$ and $t_i \in \{0, t\}$ to get an $\text{HW}(K_u[4kt]; 4k, 4kt; \sum_{i=1}^{2k(u-1)} t_i, 2kt(u-1) - \sum_{i=1}^{2k(u-1)} t_i)$. Further, applying Construction 3.3 with an $\text{HW}(4kt; 4k, 4kt; \alpha', 2kt-1-\alpha')$, $0 \leq \alpha' \leq 2kt-1$, from Theorem 1.6 in [27], we can obtain an $\text{HW}(4ktu; 4k, 4kt; \alpha' + \sum_{i=1}^{2k(u-1)} t_i, (2ktu-1) - (\alpha' + \sum_{i=1}^{2k(u-1)} t_i))$. It's easy to check that $\alpha' + \sum_{i=1}^{2k(u-1)} t_i$ can cover all the integers from 0 to $2ktu-1$. \square

5 Concluding remarks

Combining Theorems 1.3 and 1.4, we have the following open problem for $m=k$ and $n=2kt+1$.

Problem 5.1. Find a solution $\text{HW}(k(2kt+1); k, 2kt+1; \alpha, \beta)$ in the following cases:

(i) $t = 1$

- 1) $k = 5$: $\beta = \{1, 2, 3\}$;
- 2) $k = 7$: $\beta \in \{1, 2, 3, 5\}$;
- 3) $k \geq 9$: $\beta \in \{1, 2, 3, 5, 7\}$.

(ii) $t = 2$

- 1) $k = 3$ or $k \geq 11$: $\beta \in [1, 2, \dots, 2k-1] \cup \{2k+1, 2k+3\}$;
- 2) $k = 5, 7, 9$: $\beta \in \{1, 2, 3, 5, 7\}$.

(iii) $t \geq 3$

- 1) $k = 3$:
 - t is odd: $\beta = \{1, 3, 5, \dots, 3t-2\} \cup \{2, 9t-3, 9t-1\}$
 - t is even: $\beta = \{1, 3, 5, \dots, 3t+3\} \cup \{2, 9t-3, 9t-1\}$;
- 2) $k \geq 5$: $\beta \in \{1, 2, 3, 5, 7\}$.

References

- [1] P. Adams, E. J. Billington, D. E. Bryant, S.I. El-Zanati, On the Hamilton-Waterloo problem, *Graphs Combin.* **18** (2002), 31-51.
- [2] P. Adams, E. J. Billington, D. G. Hoffman and C. C. Lindner, The generalized almost resolvable cycle system problem, *Combinatorica* **30** (2010), 617-625.
- [3] B. Alspach, H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory Ser. B* **81** (2001), 77-99.
- [4] A. Assaf, A. Hartman, Resolvable group divisible designs with block size 3, *Discrete Math.* **77** (1989), 5-20.
- [5] J. Asplund, D. Kamin, M. Keranen, A. Pastine, S. Ozkan, On the Hamilton-Waterloo problem with triangle factors and C_{3x} -factors, (2015), arXiv:1510.04607 [math.CO].
- [6] B. Alspach, P. J. Schellenberg, D. R. Stinson, D. Wagner, The Oberwolfach problem and factors of uniform odd length cycles, *J. Combin. Theory Ser. A* **52** (1989), 20-43.
- [7] M. Buratti, H. Cao, D. Dai and T. Traetta, A complete solution to the existence of (k, λ) -cycle frames of type g^u , preprint.
- [8] A. Burgess, P. Danziger, T. Traetta, On the Hamilton-Waterloo problem with odd orders, (2015), arXiv:1510.07079 [math.CO].
- [9] J. Burling and K. Heinrich, Near 2-factorizations of $2K_n$: cycles of even length, *Graphs Combin.* **5**(1989), 213-221.
- [10] E. J. Billington, D. G. Hoffman, C. C. Lindner, M. Meszka, Almost resolvable minimum coverings of complete graphs with 4-cycles, *Australas. J. Combin.* **50** (2011), 73-85.
- [11] H. Cao, M. Niu, C. Tang, On the existence of cycle frames and almost resolvable cycle systems, *Discrete Math.* **311** (2011), 2220-2232.

- [12] P. Danziger, G. Quattrocchi, B. Stevens, The Hamilton-Waterloo problem for cycle sizes 3 and 4, *J. Combin. Des.* **17** (2009), 342-352.
- [13] J. H. Dinitz, A. C. H. Ling, The Hamilton-Waterloo problem with triangle-factors and Hamilton cycles: The case $n \equiv 3 \pmod{18}$, *J. Combin. Math. Combin. Comput.* **70** (2009), 143-147.
- [14] J. H. Dinitz, A. C. H. Ling, The Hamilton-Waterloo problem: The case of triangle-factors and one Hamilton cycle, *J. Combin. Des.* **17** (2009), 160-176.
- [15] I. J. Dejter, C. C. Lindner, M. Meszka, C. A. Rodger, Corrigendum/addendum to: almost resolvable 4-cycle systems, *J. Combin. Math. Combin. Comput.* **66** (2008), 297-298.
- [16] I. J. Dejter, C. C. Lindner, C. A. Rodger, M. Meszka, Almost resolvable 4-cycle systems, *J. Combin. Math. Combin. Comput.* **63** (2007), 173-181.
- [17] A. Erzurumluoğlu and C. A. Rodger, Fair holey hamiltonian decompositions of complete multipartite graphs and long cycle frames, *Discrete Math.* **338**(2015), 1173-1177.
- [18] K. Heinrich, C. C. Lindner and C. A. Rodger, Almost resolvable decompositions of $2K_n$ into cycles of odd length, *J. Combin. Theory Ser. A* **49**(1988), 218-232.
- [19] P. Horak, R. Nedela, A. Rosa, The Hamilton-Waterloo problem: The case of Hamilton cycles and triangle-factors, *Discrete Math.* **284** (2004), 181-188.
- [20] D. G. Hoffman, P. J. Schellenberg, The existence of C_k -factorizations of $K_{2n}-F$, *Discrete Math.* **97** (1991), 243-250.
- [21] D. C. Kamin, Hamilton-Waterloo problem with triangle and C_9 -factors, Master's Thesis Michigan Technological University, 2011. <https://digitalcommons/etds/207>
- [22] M. Keranen, S. Özkan, The Hamilton-Waterloo problem with 4-cycles and a singer factor of n -cycles, **29** (2013), 1827-1837.
- [23] J. Liu, A generalization of the Oberwolfach problem and C_t -factorizations of complete equipartite graphs, *J. Combin. Des.* **8** (2000), 42-49.
- [24] J. Liu, The equipartite Oberwolfach problem with uniform tables, *J. Combin. Theory Ser. A* **101** (2003), 20-34.
- [25] J. Liu, D. R. Lick, On λ -fold equipartite Oberwolfach problem with uniform table sizes, *Ann. Comb.* **7** (2003), 315-323.
- [26] H. C. Lei, H. L. Fu, The Hamilton-Waterloo problem for triangle-factors and heptagon-factors, *Graphs Combin.* **32** (2016), 271-278.
- [27] H. C. Lei, H. L. Fu, H. Shen, The Hamilton-Waterloo problem for Hamilton cycles and C_{4k} -factors, *Ars. Combin.* **100** (2011), 341-347.
- [28] C. C. Lindner, M. Meszka, A. Rosa, Almost resolvable cycle systems-an alogue of Hanani triple systems, *J. Combin. Des.* **17** (2009), 404-410.
- [29] H. C. Lei, H. Shen, The Hamilton-Waterloo problem for Hamilton cycles and triangle-factors, *J. Combin. Des.* **20** (2012), 305-316.
- [30] A. Muthusamy and A. Shanmuga Vadivu, Cycle frames of complete multipartite multi-graphs - III, *J. Combin. Des.* **22**(2014), 473-487.

- [31] M. X. Niu, H. T. Cao, More results on cycle frames and almost resolvable cycle systems, *Discrete Math.* **312** (2012), 3392-3405.
- [32] U. Odabaşı, S. Özkan, The Hamilton-Waterloo problem with C_4 and C_m factors, *Discrete Math.* **339** (2016), 263-269.
- [33] W. L. Piotrowski, The solution of the bipartite analogue of the Oberwolfach problem, *Discrete Math.* **97** (1991), 339-356.
- [34] R. Rees, Two new direct product-type constructions for resolvable group-divisible designs, *J. Combin. Des.* **1** (1993), 15-26.
- [35] D. R. Stinson, Frames for Kirkman triple systems, *Discrete Math.* **65**(1987), 289-300.
- [36] M. Šajna, Cycle decompositions: complete graphs and fixed length cycles, *J. Combin. Des.* **10** (2002), 27-78.
- [37] S. A. Vanstone, D. R. Stinson, P. J. Schellenberg, A. Rosa, R. Rees, C. J. Colbourn, M. W. Carter, J. E. Carter, Hanani triple systems, *Israel J. Math.* **83** (1993), 305-319.
- [38] D. West, Introduction to Graph Theory, 2nd Edition, Prentice Hall, 2001.
- [39] L. Wang, F. Chen, H. T. Cao, The Hamilton-Waterloo problem for C_3 -factors and C_n -factors, preprint.
- [40] L. Zhang, More results on cycle frames, Master Thesis, Nanjing Normal University, 2013.